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Bifurcations of symmetrical caustics: the creation–annihilation of two symmetric butterflies

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Abstract. The problem of the bifurcations of symmetrical singularities is attacked in the case of the most simple symmetry. We show that the Lagrangian singularities invariant through the mirror symmetry may undergo two types of bifurcations involving butterflies. In both cases, a pair of symmetric butterflies is created or disappears. In addition, we give a concrete application of our results in the domain of geometrical optics.

(Some figures in this article appear in colour in the electronic version; see www.iop.org)

1. Introduction

Symmetrical caustics are commonly encountered in many areas of physics, not only in geometrical optics [1, 2], or in wave optics [3], but also in structural mechanics [4], ballistic heat pulses in crystals [5], bound states of Hamiltonian systems [6], etc.

Mathematically, a caustic may be understood as a Lagrangian singularity [7]. The theory of the Lagrangian singularities has been successfully applied in physics, for instance, to the study of the diffraction patterns of the optical caustics [8]. However, the physical systems often present symmetries and in that case the general theory is not directly applicable to them. A specific theory of the symmetrical Lagrangian singularities actually exists but it is far from being complete and only particular cases are known [9–11]. Moreover, to our knowledge, almost nothing is known about the bifurcations of the symmetrical caustics, although they would help one to understand how the symmetrical singularities may be produced, or how they may be destroyed. In other words, the study of the bifurcations of the symmetrical singularities is important, since it allows one to understand their stability. In this paper, we study some bifurcations of a caustic invariant through a symmetry, in the 3D space. We choose a simple type of symmetry, namely mirror symmetry. We also choose a simple type of caustic having mirror symmetry: the symmetric butterfly (denoted by B_3 [12]), which is the only point singularity with minimal corank 1 [10]. The symmetric butterflies are encountered, for instance, in polyhedral caustics, around the three-fold axes [13].

The symmetric butterfly B_3 is represented in figure 1. Around it, the caustic is composed of two types of points: the fold-points (denoted by A_2) which form a self-intersecting surface and the cusp-points (denoted by A_3) which form edges of regression. There are three lines of self-intersection and four cusp-lines, all ending at the B_3-point. Two of the cusp-lines are contained in the mirror. The two other cusp-lines are mutually exchanged through the mirror symmetry. It is convenient to distinguish two parts around the symmetric butterfly. As one

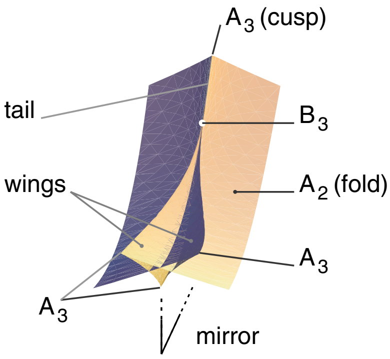

Figure 1. Around a symmetric butterfly (white point B_3), the caustic surface is composed of a self-intersecting fold-surface A_2 ending at four cusp-lines A_3: one, the *tail*, on a side of the symmetric butterfly and three on the other side, which delimit two triangular parts on the caustic, the *wings*.

moves on the mirror along the cusps, one passes from the part of the caustic located between the two symmetrical cusp-lines, the *wings*, to the part located beyond the B_3-point and which contains a unique cusp-line, the *tail* (figure 1).

The symmetric butterfly may be seen as a swallowtail A_4 lying on a cusp-line A_3. The mirror symmetry forces the swallowtail to stay on the cusp-line, a position which would be unstable in the absence of symmetry. This characterization of B_3 will be used in the next section.

2. Mathematical model of symmetric butterflies and their bifurcations

We recall that each type of caustic point, considered as a Lagrangian singularity, may be described by a family of functions Φ, named its generating family, from which all the features of the singularity may be deduced [7]. It has been proved that the generating family of the symmetric butterfly has the following form [10]:

$$\Phi(\lambda; x_1, x_2, x_3) = \lambda^6 + x_2\lambda^2 + x_3\lambda^4 + x_1\lambda. \tag{1}$$

Here λ is the argument of the function Φ and the space coordinates x_1, x_2, and x_3 are the parameters of the family of functions. The caustic surface C is the set of the points (x_1, x_2, x_3) for which there exists λ satisfying the following equations [7]:

$$\frac{\partial\Phi}{\partial\lambda} = \frac{\partial^2\Phi}{\partial\lambda^2} = 0. \tag{2}$$

In the space (λ, x_1, x_2, x_3), both equations in (2) define the set Σ of the singular points, which projects in the ordinary space (x_1, x_2, x_3) into C. Reporting expression (1) into (2) one obtains x_1 and x_2 as functions of λ and x_3 and the caustic appears as a (singular) surface parametrized by λ and x_3:

$$\begin{aligned} x_1(\lambda, x_3) &= 24\lambda^5 + 8x_3\lambda^3 \\ x_2(\lambda, x_3) &= -15\lambda^4 - 6x_3\lambda^2. \end{aligned} \tag{3}$$

Under the tranformation $\lambda \to -\lambda$, x_1 is transformed into its opposite $-x_1$, whereas x_2 remains fixed, showing that the caustic is invariant through the mirror $x_1 = 0$.

Now, the bifurcations of Lagrangian singularities evolving in time are obtained by constructing a 'big caustic' in the space–time (the product of the space (x_1, x_2, x_3) and the x_4-axis) and next by cutting it by appropriate hyper-surfaces [7]. Each section represents the instantaneous evolving caustic. In our case, we construct the big caustic by continuously translating the symmetric butterfly along the new x_4-axis. Its generating family $\Phi(\lambda; x_1, x_2, x_3, x_4)$ keeps the same form (1). We introduce the time function

$\tau(x_1, x_2, x_3, x_4) = x_4^2 - x_3$, whose level surfaces $\tau = t$ define the hyper-surfaces. The coordinate x_3 is now eliminated on the hyper-surfaces: $x_3 = x_4^2 - t$. We then obtain the time-dependent generating family

$$\Phi(\lambda; x_1, x_2, x_4) = \lambda^6 + x_2\lambda^2 + (x_4^2 - t)\lambda^4 + x_1\lambda. \tag{4}$$

The corresponding caustic, in the space (x_1, x_2, x_4), is given by

$$\begin{aligned} x_1(\lambda, x_4) &= 24\lambda^5 + 8(x_4^2 - t)\lambda^3 \\ x_2(\lambda, x_4) &= -15\lambda^4 - 6(x_4^2 - t)\lambda^2. \end{aligned} \tag{5}$$

These expressions show that the caustic is left invariant through the mirror symmetry $x_1 \to -x_1$ for all times t.

We have now to understand what is the caustic evolution represented by the equations (5), and more precisely, we have to determine whether there exist butterflies, for the different values of the time t. Since a symmetric butterfly may be viewed as a A_4-point locked on a A_3-line, the problem is reduced to the identification of the A_3 and A_4-points of the caustic. We use the method we have developed in a previous work [15] and which is based on the notion of Thom's class $\Sigma^{i_1,\ldots,i_k}$ [16]. Let us briefly recall that the Thom's class Σ^1 corresponds to the set of the regular points of the caustic, i.e. the fold-points A_2. The class $\Sigma^{1,1}$ corresponds to the cusp-points (A_3), the class $\Sigma^{1,1,1}$ to the swallowtails (A_4) and the class Σ^2 to the umbilics (either elliptic or hyperbolic). Each class is defined by a condition on the rank of some mapping. The reader is invited to refer to our paper [15] for more details about the Thom's classes and their use in geometrical optics.

The conditions defining the class $\Sigma^{1,1}$ of the cusps A_3 lead to the equations $\lambda = 0$ or $5\lambda^2 + x_4^2 = t$. The first equality shows that the x_4-axis is a cusp-line for all t. The second equality shows that for $t > 0$ there exists another cusp-line, which is closed. These two solutions allow us to find that the conditions for the class $\Sigma^{1,1,1}$ of the swallowtails take the form $\lambda = 0, x_4^2 - t = 0$. Solutions only exist for $t > 0$: $Q_1 = (0, 0, \sqrt{t})$ and $Q_2 = (0, 0, -\sqrt{t})$. They are located on the x_4-axis, which is a cusp-line. We can then conclude that our generating family (4) describes the creation (for increasing t) or the annihilation (for decreasing t) of a pair of two symmetric butterflies, at $t = 0$. The whole process is identical (isomorphic) to that studied further, in section 4.1, and it is represented in figure 2. Both B_3-points are linked by the three cusps-lines of their wings and we name this bifurcation a *wings-to-wings* bifurcation of symmetric butterflies. The closed cusp-line forms lips. They are reminiscent of the classical lips bifurcation [7], but they are much more elongated: our calculation shows that their length (x_4 coordinate) scales as $t^{1/2}$ and their width (x_1 coordinate) as $t^{5/2}$, whereas the scaling is respectively, $t^{1/2}$ and $t^{3/2}$ in the lips bifurcation.

Starting now from the time function $\tau(x_1, x_2, x_3, x_4) = -x_4^2 - x_3$, one finds, by the same type of calculation, that a pair of symmetric butterflies is created or annihilates at time $t = 0$, for respectively increasing or decreasing values of t. The process, identical to that studied in section 4.2, is represented in figure 3. The butterflies are now oriented in the opposite direction along the x_4-axis and they are linked by their tails. So we name this bifurcation the *tail-to-tail* bifurcation of symmetric butterflies. The symmetrical cusp-lines undergo a transformation which is remisniscent of the *bec à bec* (beak-to-beak) bifurcation [7], but with much more elongated beaks, since the length and the width scale as $t^{1/2}$ and $t^{5/2}$, instead of $t^{1/2}$ and $t^{3/2}$ for the *bec à bec* bifurcation.

Other bifurcations of symmetrical butterflies may exist. The general problem of classifying all the possible bifurcations of these symmetrical caustics is a very complicated and challenging task. In this paper our aim was to solve what we believe constitutes the simplest case.

At this stage, two mathematical models of bifurcations of symmetrical Lagrangian singularities have beeen obtained. However, we now have to specify the domain of application

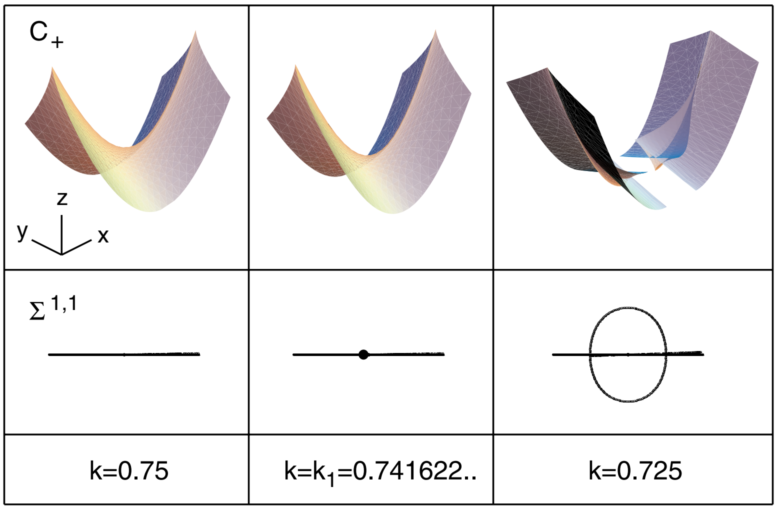

Figure 2. Wings-to-wings bifurcation. Sketch of the caustic C_+ in the physical space (x, y, z) (top) and of the set $\Sigma^{1,1}$ in the source space (bottom), for three values of the bifurcation parameter k: above the bifurcation point k_1 ($k = 0.75$), at k_1 (=0.741 622 . . .) and below k_1 ($k = 0.725$). In the latter case, the details are shown by cutting the caustic surface and by pulling apart both parts.

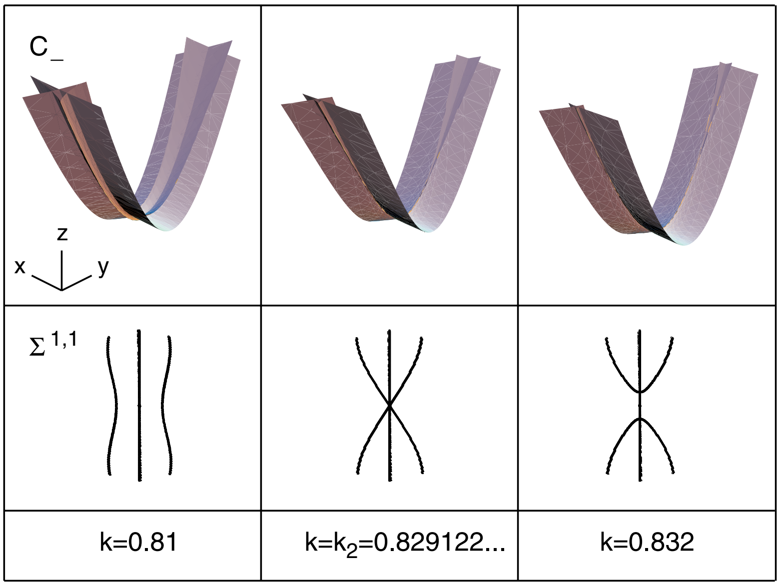

Figure 3. Tail-to-tail bifurcation. Sketch of the caustic C_- in the physical space (x, y, z) (top) and of the set $\Sigma^{1,1}$ in the source space (bottom), for three values of the bifurcation parameter k: below the bifurcation point k_2 ($k = 0.81$), at k_2 (=0.829 122 . . .) and above k_2 ($k = 0.832$).

of our results to the physics. Indeed, it is known that the physical laws may strongly affect the stability of the singularities, that is to say the conditions of their bifurcations. For example, four of the 11 bifurcations of (non-symmetrical) Lagrangian singularities are excluded in geometrical optics, because of a convexity property imposed by the eiconal equation [14]. It is then necessary to check that both bifurcations we have found are effectively realized in a physical model. To this end, we shall choose an optical model in the following and we shall prove that its singularities undergo both bifurcations when some parameter is varied.

3. The optical model

Our physical model is a wavefront propagating in a homogeneous and isotropic medium of refractive index 1. As in a previous work [13], we define the initial wavefront W with the required symmetry by introducing a function V: $W = \{(x, y, z), V(x, y, z) = k\}$, where k is some constant. More precisely we fix n points P_i in R^3 and n masses m_i. The function V is given by

$$V(P) = \sum_{i=1}^{n} m_i d(P, P_i) \tag{6}$$

where $d(P, P_i)$ denotes the distance from P to P_i. We set $m_i = 1/n$ for all i. The model is defined by six points: four points $P_1 = (a, b, 0)$, $P_2 = (a, -b, 0)$, $P_3 = (-a, b, 0)$ and $P_4 = (-a, -b, 0)$ forming a rectangle in the plane $z = 0$, and two points $P_5 = (0, 0, c)$, $P_6 = (0, 0, -c)$ lying on the z-axis. The rays are the normals to W and the caustic C is the envelope of the rays. From now on, we shall restrict our attention to a small part of W, centred around its point $R = (0, 0, z_0 = -k + 2\sqrt{k^2 - a^2/3 - b^2/3})$. Through R pass two planes of symmetry: $x = 0$ and $y = 0$. As a consequence, each of these planes contains a cusp-line. The symmetry around R is the rectangular symmetry $2mm$ (also denoted by C_{2v}). This symmetry, richer than the simple symmetry m needed for the appearance of symmetric butterflies, is useful to obtain, as we shall see, both bifurcations in the same model.

The constants a, b and c are now fixed: $a = 0.7$, $b = 0.5$ and $c = 0.1$. The bifurcation parameter of the model is the parameter k, on which depend the surface W and the caustic C.

A point $Q = (X, Y, Z)$ of the congruence of the rays is determined by three coordinates: the coordinates x, y of the origin on W of the ray passing through Q, and the coordinate s along the ray. We have

$$\begin{aligned} X(x, y, s) &= x + sV_x \\ Y(x, y, s) &= y + sV_y \\ Z(x, y, s) &= z_1(x, y) + sV_z \end{aligned} \tag{7}$$

where V_x stands for $\partial V/\partial x(x, y, z_1(x, y))$, etc, and where $z_1(x, y)$ is defined implicitly by the equation of W: $V(x, y, z_1(x, y)) = k$. The relation (7) defines a mapping f from the source space (x, y, s) into the physical space (X, Y, Z). The singular set Σ is the set of the points of the source space where f has a rank $\mathrm{rk}(f)$ less than its maximal possible value 3. The caustic C is the image of Σ by f: $C = f(\Sigma)$.

Here too, we have to determine the different types of the caustic points. The class Σ^1 of the fold-points is the set of the points where f has a rank equal to 2. Its equation is given by

$$\det \partial(X, Y, Z)/\partial(x, y, s) = 0. \tag{8}$$

Equation (8) involves partial derivatives of z_1, which may be obtained from the basic datum V of the problem by derivating the relation $V(x, y, z_1(x, y)) = k$. More precisely, putting $\mathcal{X}(x, y) = V_x(x, y, z_1(x, y))$, etc, the equation for s is

$$a_0 + a_1 s + a_2 s^2 = 0 \tag{9}$$

where

$$\begin{aligned} a_0 &= \mathcal{Z} - z_1\mathcal{X} - z_1\mathcal{Y} \\ a_1 &= \mathcal{Z}(\mathcal{X}_x + \mathcal{Y}_y) + \mathcal{X}(\mathcal{Y}_x z_{1y} - \mathcal{Y}_y z_{1x} - \mathcal{Z}_x) + \mathcal{Y}(\mathcal{X}_y z_{1x} - \mathcal{X}_x z_{1y} - \mathcal{Z}_y) \\ a_2 &= \mathcal{Z}(\mathcal{X}_x\mathcal{Y}_y - \mathcal{X}_y\mathcal{Y}_x) + \mathcal{X}(\mathcal{Y}_x\mathcal{Z}_y - \mathcal{Y}_y\mathcal{Z}_x) + \mathcal{Y}(\mathcal{Z}_x\mathcal{X}_y - \mathcal{Z}_y\mathcal{X}_x). \end{aligned} \tag{10}$$

Equation (9) has two solutions $s_\pm = (-a_1 \pm \sqrt{a_1^2 - 4a_0a_2})/2a_2$, showing that the singular set Σ is composed of two sheets Σ_+ and Σ_-. Consequently, the caustic $C = f(\Sigma)$ is also composed of two sheets $C_+ = f(\Sigma_+)$ and $C_- = f(\Sigma_-)$.

By reporting the solution s into (7) we define the restriction of f to Σ^1. The class $\Sigma^{1,1}$, associated with the set of the cusp-lines $f(\Sigma^{1,1})$, is defined by the condition that this restriction has a rank equal to 1.

The equations for Σ^1 and $\Sigma^{1,1}$ are very intricate, but they may be solved numerically.

4. The two bifurcations

We numerically calculate the two Thom's classes Σ^1 and $\Sigma^{1,1}$ giving the fold-surface $f(\Sigma^1)$ and the set of the cusp-lines $f(\Sigma^{1,1})$ for different values of k, by starting from the value $k_0 = 0.8$. For all the values of k, the x- and y-axes of the source space are in $\Sigma^{1,1}$, but located on a different sheet, Σ_+ for the x-axis and Σ_- for the y-axis. In the physical space they correspond to two (not intersecting) cusp-lines, each one being located in a mirror plane.

4.1. First bifurcation

The value of k is at first progressively decreased from k_0. We consider here the sheet C_+, which contains a cusp-line in the plane of symmetry (x, z). Up to $k_1 = 0.741\,623\ldots$, the cusp-line remains stable (figure 2, left). At this k_1 value a degenerate point appears on the cusp-line (it cannot be identified on the drawing of the caustic) (figure 2, middle). Next, a pair of symmetric butterflies is formed in the plane of symmetry (x, z) (figure 2, right). The relative distance between the butterflies increases for decreasing values of k. Because of the presence of the second mirror, the set composed of the two butterflies is also symmetrical with respect to the plane (y, z). From each butterfly two new cusp-lines start and next converge to the other butterfly. In this transformation we recognize a wings-to-wings bifurcation of symmetric butterflies. In the source space, the initial cusp-line is associated with a straight line and the two new cusp-lines with a small ellipse (figure 2). The butterflies correspond to the intersection of the straight line and the ellipse. The ellipse is reduced to a single point at $k = k_1$. This characterization of the bifurcation point allows us to numerically obtain k_1 with a relative error of less than 10^{-6}.

4.2. Second bifurcation

The value of k is now progressively increased from k_0. We consider the sheet C_-, which contains a cusp-line in the plane of symmetry (y, z). This cusp-line is surrounded by two other cusp-lines, which mutually exchange through the mirror symmetry (figure 3, left). With increasing k, these two cusp-lines become pinched and get closer to the third one. The pinching increases until the three cusp-lines come into contact for $k = k_2 = 0.829\,122\ldots$ (figure 3, middle). Next, two symmetric butterflies are formed in the plane of symmetry (y, z) (figure 3, right) and their relative distance increases with k. They are linked by one cusp-line. Both initial symmetrical cusp-lines have been cut and reconnected at the B_3-points. Here we recognize the tail-to-tail bifurcation of symmetric butterflies. In the source space, the initial cusp-lines correspond, locally near the bifurcation point, to two arcs of hyperbolas (see figure 3). At the bifurcation point $k = k_2$, both arcs connect and form a cross. This characterization of the bifurcation point is used to obtain k_2 with the same relative error 10^{-6}.

Since the cusp-lines in the source space (set $\Sigma^{1,1}$) form an ellipse in the wings-to-wings bifurcation and an hyperbola in the tail-to-tail bifurcation, both bifurcations may be considered as a pair of two dual bifurcations. However, the tail-to-tail bifurcation is distinguished by the fact that, in contrast to the wings-to-wings bifurcation, it has a global character, since it modifies the connectivity of the network of the cusp-lines.

5. Conclusion

We have studied bifurcations of Lagrangian singularities having the mirror symmetry. We have shown that these singularities may bifurcate through the creation or the annihilation of a pair of symmetric butterflies contained in the mirror. There exist two types of such bifurcations, each one being distinguished by the relative orientations of the butterflies: either wings-to-wings or tail-to-tail. We have checked that both bifurcations are realized in geometrical optics. However, the general (Lagrangian) nature of our results suggests that they may also be applied to other physical systems, such as mechanical systems or waves.

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